# *Kelly bet sizing*[1]

## *—Humility Consulting*

*May 2, 2015*

[1] as discussed in p. 74 of
Aaron Brown. *Red-Blooded Risk*. Wiley Finance, 2011

This document derives Kelly's rule for optimal bet sizing.

Here's the game. You have a $10,000 stake. You face a series of 60 wins and 40 losses. Their order is chosen by me. You choose each bet's size. After you tell me the size, I get to spend, or not, one of my 40 kills—if I have any left. Your win is my loss, and vice-versa. After 100 rounds, we're done.

If you always bet the same fixed percentage of your stake, then[2] I have no strategic role to play in determining your overall takeaway. I can't affect your outcome if you play that way.

[2] since multiplication is commutative

## *The Math*

I'll denote by $a$ the aggressiveness of your bets. If you win a bet then $\texttt{stake}_{t+1} = \texttt{stake}_t \cdot (1+a)$ and if you lose a bet then $\texttt{stake}_{t+1} = \texttt{stake}_t \cdot (1-a)$. A win, then a loss, turns out the same as a loss, then a win:

$$\begin{aligned}\texttt{stake}_{t+2} &= (1+a)\cdot(1-a)\cdot\texttt{stake}_t \\ &= (1-a)\cdot(1+a)\cdot\texttt{stake}_t.\end{aligned}$$

In these terms, Kelly's question is: what is the optimal[3] aggressiveness $a^*$ to **maximise your final takeaway**[4]?

[3] assuming, to remove my strategic input, that it's the same fraction $a$ every time

[4] i.e., $\arg\max_{\{a\}} \texttt{stake}_{100} \stackrel{def}{=} a^*$

## *The Answer*

> Twenty percent is the perfect amount to bet. Betting a higher or a lower fraction means doing worse. … While there are people who dislike the Kelly criterion for various reasons, no intelligent person disputes this aspect of the result.
>
> —Brown, p. 76

http://books.google.com/books?id=eu925DG2xeYC&lpg=PP1&dq=red-blooded%20risk&pg=PT60#v=snippet&q=mathematical&f=false

*Solving for optimal aggressiveness*

This can be solved with calculus 101. Since the order doesn't matter and there's a fixed number of wins and losses, we can write the following formula for your total winnings at 60/40 odds:

$$\$1000 \cdot (1+a)^{60} \cdot (1-a)^{40} \qquad \text{(take)}$$

.

To find the optimal aggressiveness, set the derivative of equation (take), with respect to $a$, equal to zero.[5] The symbolic derivative of

$$(1+a)^{60} \cdot (1-a)^{40}$$

with respect to $a$ is:

$$60 \cdot (1+a)^{59} \cdot (1-a)^{40} \quad - \quad 40 \cdot (1+a)^{60} \cdot (1-a)^{39} \qquad \text{(D)}$$

. Setting (D) = 0 tells me a property that is true of the optimal $a^*$. Moving things around, that property can be restated as:

$$60 \cdot \frac{(1-a^*)^{40}}{(1-a^*)^{39}} \quad = \quad 40 \cdot \frac{(1+a^*)^{60}}{(1+a^*)^{59}}$$

which reduces to the much nicer

$$60 \cdot (1-a^*) \quad = \quad 40 \cdot (1+a^*).$$

Solving then for the optimal aggressiveness $a^*$:

$$60 - 60a^* = 40 + 40a^* \qquad (1)$$

$$20 = 100a^* \qquad (2)$$

which is what Brown gets: 20% = $\frac{1}{5}$ for the optimal $a^*$ against these odds.

[5] (There are further conditions to make sure this works, which I'm leaving out.)

The product rule says $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$.

Thanks, Artemy!

*Brown's conclusion*

The "Ed Thorp takeaway" is that risk management does not mean taking no risks. You can't sit on your hands. Bill Gross says to "avoid portfolio mush", which is similar & related.

Contrast this to "lazy CAPM" style thinking. When you have an edge (like a 60/40 edge) you need to exploit it. Betting less than 20% of your stake against 60/40 odds is[6] suboptimal. Less aggressive betting does not move you along an optimal frontier; it moves you *off* the optimal frontier.

[6] In this model.

*References*

Aaron Brown. *Red-Blooded Risk*. Wiley Finance, 2011.